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MEMORANDUM

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SIMPLEX METHOD AND THEORY
Notes on Linear Programming
and Extensions - Part 62

A. W. Tucker



PREPARED FOR:

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The **RAND** Corporation
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SIMPLEX METHOD AND THEORY
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PREFACE

Part of the RAND research program consists of basic support studies in mathematics. One aspect of this, of very considerable applicability in large governmental, military, and industrial operations, is concerned with linear programming.

In the present Memorandum the author discusses the basic structure of this theory.

This Memorandum will appear as part of a book, Symposium on Mathematical Optimization Techniques to be published by the University of California Press.

SUMMARY

In this Memorandum, the author discusses the simplex method of linear programming in a format designed to exhibit over-all structure rather than specific operational details. Various terminal possibilities are represented schematically and geometrically, and it is shown that transposition duality theorems can be regarded as corollaries of the duality theorem for a homogeneous linear program.

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SIMPLEX METHOD AND THEORY

1. INTRODUCTION

The simplex method (1947) of George B. Dantzig [1] is much more than the basic computational tool of linear programming. It is a combinatorial algorithm that provides constructive means of establishing fundamental theorems of linear programming [2]—as well as like theorems in cognate areas, such as von Neumann's Minimax Theorem for matrix games [3] and Farkas' Theorem for linear inequalities. Its characteristic pivot transformations are related in an essential way to Gauss-Jordan elimination [4] and to a combinatorial equivalence of matrices [5].

This paper discusses the simplex method in a format designed to exhibit over-all structure rather than specific operational details. The various terminal possibilities are represented schematically and geometrically. Also, it is shown that transposition duality theorems [6], such as the classical ones of Gordan, Farkas, Stiemke and Motzkin, can be regarded as corollaries of the duality theorem for a "homogeneous linear program."

The schemata and block-pivot transformations used in this paper seem to be important methodological devices. They follow closely along lines developed by the author in a previous paper concerned with solutions of matrix games by linear programming [7].

$$(2.3) \quad \left. \begin{array}{l} -a_{11}y_1 - a_{12}y_2 - \dots - a_{1n}y_n = x_1 \\ -a_{21}y_1 - a_{22}y_2 - \dots - a_{2n}y_n = x_2 \\ \vdots \quad \quad \quad \vdots \\ -a_{m1}y_1 - a_{m2}y_2 - \dots - a_{mn}y_n = x_m \end{array} \right\} -AY = X.$$

These two systems are dual in the sense that the inner (scalar) product satisfies the equation

$$(2.4) \quad [\Xi, H] \begin{bmatrix} X \\ Y \end{bmatrix} = \Xi X + HY = \Xi(-AY) + (\Xi A)Y = 0$$

for any Ξ, H satisfying the column system (2.2) and any X, Y satisfying the row system (2.3).

The column system (2.2) consists of n linear equations in $m + n$ variables; these n equations are linearly independent because each η occurs with nonzero coefficient in just one equation. The row system (2.3) consists of m linear equations in $n + m$ variables; these m equations are linearly independent because each x occurs with nonzero coefficient in just one equation. If the Greek variables Ξ, H are regarded as (row) coordinates in a space of $m + n$ dimensions and the Latin variables X, Y as (column) coordinates in the same space, then the solution sets of (2.2) and (2.3) are linear subspaces of complementary dimensions m and n , respectively, in the space of $m + n$ dimensions. Because of (2.4), these are complementary orthogonal linear subspaces. Thus the "duality" of linear systems has the geometric interpretation of "orthogonal complementarity."

3. BLOCK-PIVOT TRANSFORMATION

Let A_{11} be a nonsingular square submatrix of A , and A_{12} , A_{21} , A_{22} the remaining submatrices of A . Then the schema (2.1) can be rewritten as

$$(3.1) \quad \begin{array}{cc|cc} & -Y_1 & -Y_2 & & \\ \hline \varepsilon_1 & A_{11} & A_{12} & & = X_1 \\ \hline \varepsilon_2 & A_{21} & A_{22} & & = X_2 \\ \hline & = H_1 & = H_2 & & \end{array}$$

Since A_{11}^{-1} exists, the subsystems

$$\varepsilon_1 A_{11} + \varepsilon_2 A_{21} = H_1 \quad \text{and} \quad -A_{11}Y_1 - A_{12}Y_2 = X_1$$

can be solved for ε_1 and Y_1 to obtain

$$\varepsilon_1 = H_1 A_{11}^{-1} - \varepsilon_2 A_{21} A_{11}^{-1} \quad \text{and} \quad Y_1 = -A_{11}^{-1} X_1 - A_{11}^{-1} A_{12} Y_2.$$

Substitution for ε_1 and Y_1 in the subsystems

$$\varepsilon_1 A_{12} + \varepsilon_2 A_{22} = H_2 \quad \text{and} \quad -A_{21}Y_1 - A_{22}Y_2 = X_2$$

yields

$$H_1 A_{11}^{-1} A_{12} + \varepsilon_2 (A_{22} - A_{21} A_{11}^{-1} A_{12}) = H_2$$

and

$$A_{21} A_{11}^{-1} X_1 - (A_{22} - A_{21} A_{11}^{-1} A_{12}) Y_2 = X_2.$$

These results are exhibited by the column and row system of the schema

$$(3.2) \quad \begin{array}{cc} & \begin{array}{cc} -X_1 & -Y_2 \end{array} \\ \begin{array}{c} H_1 \\ \Xi_2 \end{array} & \begin{array}{|cc|} \hline A_{11}^{-1} & A_{11}^{-1}A_{12} \\ \hline -A_{21}A_{11}^{-1} & A_{22}-A_{21}A_{11}^{-1}A_{12} \\ \hline \end{array} \begin{array}{c} =Y_1 \\ =X_2 \end{array} \\ & \begin{array}{cc} = \Xi_1 & = H_2 \end{array} \end{array}$$

The schema (3.2) is equivalent to the schema (3.1) in the sense that the column equation systems of (3.1) and (3.2) have the same solutions Ξ, H and the row equation systems of (3.1) and (3.2) have the same solutions X, Y .

Let r be the order of the nonsingular square submatrix A_{11} , the choice of which determines uniquely the transformation from the schema (3.1) to the equivalent schema (3.2). Then the transformation from (3.1) to (3.2) is called a block-pivot transformation of order r , the nonsingular square submatrix A_{11} of order r being called the block-pivot. It can readily be verified that the inverse of the block-pivot transformation from (3.1) to (3.2) is a block-pivot transformation from (3.2) to (3.1), the block-pivot being A_{11}^{-1} .

Any nonzero entry of the matrix A determines a block-pivot A_{11} of order one; the corresponding pivot transformation of order one is called an elementary pivot transformation.

Elementary pivoting, utilized so effectively in the simplex method, has its roots in the classical process of Gauss-Jordan (complete) elimination.

Note that the block-pivot transformation of order r from (3.1) to (3.2) exchanges r of the individual marginal labels at the left with r labels at the bottom and r parallel labels at the right with r parallel labels at the top, signs being reversed in the latter exchange. Such a block-pivot transformation can always be decomposed into a succession of elementary pivot transformations, exchanging just one label on a margin at a time; conversely, any finite succession of elementary pivot transformations is summarized by a single block-pivot transformation (as explained in [5] and illustrated in [7]).

The m by n matrices in (3.1) and (3.2), or any row and/or column permutations thereof, are combinatorially equivalent in a sense discussed by the author in [5].

4. DUAL LINEAR PROGRAMS

Here the format developed in the two previous sections will be used, with some change of symbols, to discuss dual linear programs.

The schema

$$(4.1) \quad \begin{array}{cccccc} & -x_1 & -x_2 & \dots & -x_N & 1 \\ \lambda_1 & a_{11} & a_{12} & \dots & a_{1N} & b_1 & = 0 \\ \lambda_2 & a_{21} & a_{22} & \dots & a_{2N} & b_2 & = 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \lambda_M & a_{M1} & a_{M2} & \dots & a_{MN} & b_M & = 0 \\ 1 & c_1 & c_2 & \dots & c_N & d & = w \\ & =\xi_1 & =\xi_2 & \dots & =\xi_N & =\omega & \end{array}$$

exhibits row and column equation systems

$$\left. \begin{array}{l} -AX + B = 0 \\ -CX + d = w \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{l} \Lambda A + C = \Xi \\ \Lambda B + d = \omega \end{array} \right.$$

which pertain to the following pair of linear programs:

(4.2) Primal Program. To maximize $w = d - CX$
constrained by $AX = B, X \geq 0$.

(4.3) Dual Program. To minimize $\omega = d + \Lambda B$
constrained by $\Lambda A + C = \Xi \geq 0$.

(A vector inequality holds for each component—i.e.,
 $X \geq 0$ means $x_1 \geq 0, x_2 \geq 0, \dots, x_N \geq 0$.) The "parameters"
 $(\lambda_1, \lambda_2, \dots, \lambda_M) = \Lambda$ in the Dual Program are unrestricted
in sign.

Let A_{11} be a nonsingular square submatrix of the matrix A above. Then the schema (4.1) can be recast as

$$(4.4) \quad \begin{array}{c} \Lambda_1 \\ \Lambda_2 \\ 1 \end{array} \begin{array}{c} -X_1 \quad -X_2 \quad 1 \\ \boxed{\begin{array}{ccc} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & d \end{array}} \\ \begin{array}{ccc} =\varepsilon_1 & =\varepsilon_2 & =w \end{array} \end{array} \begin{array}{l} = 0 \\ = 0 \\ = w \end{array}$$

(Of course, the Λ_2 -headed row in (4.4) will be vacuous if the submatrix A_{11} omits no row of A , and the X_2 -headed column in (4.4) will be vacuous if A_{11} omits no column of A .)

$$\begin{aligned} \bar{A}_{11} &= A_{11}^{-1}, & \bar{A}_{12} &= A_{11}^{-1}A_{12}, & \bar{B}_1 &= A_{11}^{-1}B_1, \\ \bar{A}_{21} &= -A_{21}A_{11}^{-1}, & \bar{A}_{22} &= A_{22} - A_{21}A_{11}^{-1}A_{12}, & \bar{B}_2 &= B_2 - A_{21}A_{11}^{-1}B_1, \\ \bar{C}_1 &= -C_1A_{11}^{-1}, & \bar{C}_2 &= C_2 - C_1A_{11}^{-1}A_{12}, & \bar{d} &= d - C_1A_{11}^{-1}B_1. \end{aligned}$$

Then the schema

$$(4.5) \quad \begin{array}{c} \varepsilon_1 \\ \Lambda_2 \\ 1 \end{array} \begin{array}{c} 0 \quad -X_2 \quad 1 \\ \boxed{\begin{array}{ccc} \bar{A}_{11} & \bar{A}_{12} & \bar{B}_1 \\ \bar{A}_{21} & \bar{A}_{22} & \bar{B}_2 \\ \bar{C}_1 & \bar{C}_2 & \bar{d} \end{array}} \\ \begin{array}{ccc} =\Lambda_1 & =\varepsilon_2 & =w \end{array} \end{array} \begin{array}{l} = X_1 \\ = 0 \\ = w \end{array}$$

results from the schema (4.4) by the block-pivot transformation having A_{11} as block-pivot.

The new schema (4.5) is equivalent to the old schema (4.4). That is, the row equation system of one schema is satisfied by any X , w satisfying the row equation system of the other schema, and the column equation system of one schema is satisfied by any Λ , $\bar{\epsilon}$, ω satisfying the column equation system of the other schema. Hence the Primal Program (4.2) calls now for maximizing w subject to the row equation system of (4.5) and the inequalities $X_1 \geq 0$, $X_2 \geq 0$, and the Dual Program (4.3) calls now for minimizing ω subject to the column equation system of (4.5) and the inequalities $\bar{\epsilon}_1 \geq 0$, $\bar{\epsilon}_2 \geq 0$.

If the schema (4.5) is such that

$$(4.6) \quad \bar{A}_{22} = 0, \bar{B}_2 = 0 \text{ (or are vacuous)}$$

and

$$(4.7) \quad \bar{B}_1 \geq 0, \bar{C}_2 \geq 0,$$

then optimal (basic) solutions of the Primal and Dual Programs, (4.2) and (4.3), can be read directly from (4.5) by setting variables at top and left margins equal to zero. These optimal solutions are

$$X_1 = \bar{B}_1 (\geq 0), X_2 = 0; w = \bar{d}$$

and

$$\Lambda_1 = \bar{c}_1, \Lambda_2 = 0; \bar{e}_1 = 0, \bar{e}_2 = \bar{c}_2 (\geq 0); \omega = \bar{d}.$$

That \bar{d} is the maximal w follows from $\bar{c}_2 \geq 0$, because

$$w = \bar{d} - \bar{c}_2 x_2 \leq \bar{d} \text{ for all } x_2 \geq 0;$$

and that \bar{d} is the minimal ω follows from $\bar{e}_1 \geq 0$, because

$$\omega = \bar{d} + \bar{e}_1 \bar{b}_1 \geq \bar{d} \text{ for all } \bar{e}_1 \geq 0.$$

The Dantzig simplex method, starting from an initial "presentation" of the pair of linear programs (4.2) and (4.3), employs a finite succession of elementary pivot transformations to achieve, if possible, a terminal "re-presentation" corresponding to a schema (4.5) for which (4.6) and (4.7) hold.

5. CANONICAL REPRESENTATION

A canonical representation ("re-presentation") of the pair of linear programs (4.2) and (4.3) is provided by any schema

$$(5.1) \quad \begin{array}{c} \varepsilon_1 \\ \Lambda_2 \\ 1 \end{array} \begin{array}{|c|c|c|} \hline 0 & -x_2 & 1 \\ \hline \bar{A}_{11} & \bar{A}_{12} & \bar{B}_1 \\ \hline \bar{A}_{21} & 0 & 0 \\ \hline \bar{C}_1 & \bar{C}_2 & \bar{d} \\ \hline \end{array} \begin{array}{l} = x_1 \\ = 0 \\ = w \end{array}$$

$$\begin{array}{ccc} = \Lambda_1 & = \varepsilon_2 & = w \end{array}$$

for which (4.6) holds. To have $\bar{A}_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12} = 0$, it is necessary and sufficient that the order of the block-pivot A_{11} equal the rank m of the matrix A , since then

$$[A_{21}, A_{22}] = A_{21}A_{11}^{-1}[A_{11}, A_{12}] .$$

If $\bar{A}_{22} = 0$, then $\bar{B}_2 = 0$ also, unless $AX = B$ is an inconsistent system of linear equations.

A partly reduced canonical schema

$$(5.2) \quad \begin{array}{c} \varepsilon_1 \\ 1 \end{array} \begin{array}{|c|c|c|} \hline 0 & -x_2 & 1 \\ \hline \bar{A}_{11} & \bar{A}_{12} & \bar{B}_1 \\ \hline \bar{C}_1 & \bar{C}_2 & \bar{d} \\ \hline \end{array} \begin{array}{l} = x_1 \\ = w \end{array}$$

$$\begin{array}{ccc} = \Lambda_1 & = \varepsilon_2 & = w \end{array}$$

results from (5.1) through deletion of the Λ_2 -headed row in (5.1). The schema (5.2) contains the same information as (5.1) with redundant parameters Λ_2 eliminated.

A fully reduced canonical schema

$$(5.3) \quad \begin{array}{cc|cc} & -x_2 & & 1 \\ \hline \varepsilon_1 & \bar{A}_{12} & & \bar{B}_1 \\ \hline 1 & \bar{C}_2 & & \bar{d} \\ \hline & =\varepsilon_2 & & =w \end{array} \quad \begin{array}{l} = x_1 \\ = w \end{array}$$

results through further deletion of the 0-headed column of (5.2). The schema (5.3) contains all the parameter-free information in (5.1) or (5.2).

If $\bar{B}_1 \geq 0$ in the above schemata, the canonical representation is primal feasible with $x_1 = \bar{B}_1 (\geq 0)$, $x_2 = 0$ yielding a feasible (basic) solution of the Primal Program (4.2). If $\bar{C}_2 \geq 0$ in the above schemata, the canonical representation is dual feasible with $\Lambda_1 = \bar{C}_1$, $\Lambda_2 = 0$ and $\varepsilon_1 = 0$, $\varepsilon_2 = \bar{C}_2 (\geq 0)$ yielding a feasible (basic) solution of the Dual Program (4.3). If both $\bar{B}_1 \geq 0$ and $\bar{C}_2 \geq 0$, the canonical representation is optimal with the above-stated feasible (basic) solutions as optimal (basic) solutions of the Primal and Dual Programs.

6. GEOMETRIC INTERPRETATION

Let the matrix A in schema (4.1) have rank m and let the number of columns of A be $N = m + n$. Let $[A, B]$ also have rank m , so that $AX = B$ is a consistent system of linear equations. Let S be a space of $N = m + n$ dimensions with a specified coordinate system, so that there is a one-to-one coorespondence between points

(or vectors) of S and ordered coordinate N -tuples, written as $\xi_1, \xi_2, \dots, \xi_N$ for row usage and as x_1, x_2, \dots, x_N for column usage. Then the solution sets

$$P = \{\Xi | \Xi = \Lambda A + C, \text{ all } \Lambda\} \quad \text{and} \quad Q = \{X | AX = B\}$$

are linear manifolds of complementary dimensions m and n in the space S . Let $\Xi = \Lambda A + C$ and $\Xi' = \Lambda' A + C$ be any two points of P , and X and X' any two points of Q . Then the equation

$$\begin{aligned} (\Xi' - \Xi)(X' - X) &= (\Lambda' A - \Lambda A)(X' - X) = \\ &= (\Lambda' - \Lambda)(AX' - AX) = 0 \end{aligned}$$

shows that P and Q are complementary orthogonal linear manifolds in S .

Let

$$R = \{\Xi | \Xi \geq 0\} = \{X | X \geq 0\}$$

be the nonnegative orthant in S . Then the feasible solution sets

$$\{\Xi | \Xi = \Lambda A + C, \Xi \geq 0\} \quad \text{and} \quad \{X | AX = B, X \geq 0\}$$

of the Dual and Primal Programs (4.3) and (4.2) are the polyhedral convex sets $P \cap R$ and $Q \cap R$, respectively.

In a canonical schema (5.3) the complementary orthogonal linear manifolds P and Q are represented by equation systems in the following "slope-intercept" form,

$$(6.1) \quad P: \quad \varepsilon_2 = \varepsilon_1 \bar{A}_{12} + \bar{C}_2$$

and

$$(6.2) \quad Q: \quad \varepsilon_1 = \varepsilon_2 (-\bar{A}_{12}^T) + \bar{B}_1^T,$$

the latter being obtained by transposing $X_1 = -\bar{A}_{12}X_2 + \bar{B}_1$ and substituting ε_1 and ε_2 for X_1^T and X_2^T . In (6.1) the m by n matrix \bar{A}_{12} is the " ε_2 : ε_1 -slope" of P (with ε_2 as "rise" and ε_1 as "run") and \bar{C}_2 is the " ε_2 -intercept" of P. In (6.2) the negative-transpose matrix $-\bar{A}_{12}^T$ is the " ε_1 : ε_2 -slope" of Q (with ε_1 as "rise" and ε_2 as "run") and \bar{B}_1^T is the " ε_1 -intercept" of Q. This canonical "slope-intercept" representation of P and Q, introduced by the author in [8], is illustrated in Fig. 1.

Let \bar{p} and \bar{q} be the intercept points (vectors)

$$\varepsilon_1 = 0, \varepsilon_2 = \bar{C}_2 \quad \text{and} \quad \varepsilon_1 = \bar{B}_1^T, \varepsilon_2 = 0$$

determined by (6.1) and (6.2). Note that the inner (scalar) product

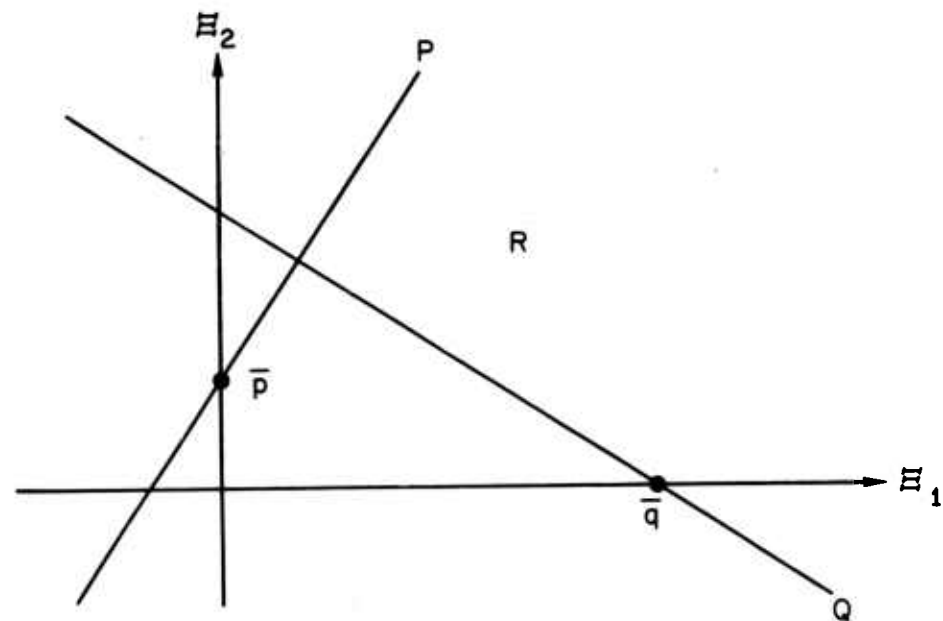


Fig.1 — Canonical slope representation of complementary orthogonal linear manifolds

$$\bar{p} \cdot \bar{q} = [0, \tau_2] \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} = 0.$$

As canonically represented in schema (5.3), the Dual Program is to minimize

$$\omega = \bar{d} + \varepsilon_1 \bar{B}_1 = \bar{d} + [\varepsilon_1, \varepsilon_2] \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} = \bar{d} + p \cdot \bar{q}$$

for p in $P \cap R$, and the Primal Program is to maximize

$$w = \bar{d} - \tau_2 x_2 = \bar{d} - [0, \tau_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \bar{d} - \bar{p} \cdot q$$

for q in $Q \cap R$. If \bar{p} belongs to $P \cap R$ and \bar{q} belongs to $Q \cap R$, then $p \cdot \bar{q} \geq 0$ for every p in $P \cap R$ and $\bar{p} \cdot q \geq 0$ for every q in $Q \cap R$ (since any two vectors in R have a non-negative inner product). Hence, since $\bar{p} \cdot \bar{q} = 0$, it is clear that

$$\omega = \bar{d} + p \cdot \bar{q} \geq \bar{d} + \bar{p} \cdot \bar{q} = \bar{d} \quad \text{for every } p \text{ in } P \cap R$$

and that

$$w = \bar{d} - \bar{p} \cdot q \leq \bar{d} - \bar{p} \cdot \bar{q} = \bar{d} \quad \text{for every } q \text{ in } Q \cap R.$$

That is, the desired minimum and maximum are attained at $p = \bar{p}$ and $q = \bar{q}$ if these points both belong to R . (The intercept points \bar{p} or \bar{q} which belong to R are the extreme points of the polyhedral convex set $P \cap R$ or $Q \cap R$.)

In summary, this geometric interpretation of a pair of linear programs (4.2) and (4.3) involves complementary orthogonal linear manifolds P and Q in a space S with nonnegative orthant R . If $P \cap R$ is nonvacuous, the Dual Program is feasible; if $Q \cap R$ is nonvacuous, the Primal Program is feasible. A canonical representation of these programs involves a joint "slope-intercept" representation of P and Q . The resulting intercept points \bar{p} and \bar{q} yield optimal solutions if they both belong to R .

7. SIMPLEX METHOD; TERMINAL POSSIBILITIES

Let $AX = B$ have a solution $X \geq 0$, i.e., $Q \cap R$ is nonvacuous and the Primal Program (4.2) is feasible. Then a proof of the simplex method, such as [9], demonstrates the existence of a finite succession of elementary pivot transformations that terminates in a canonical representation for which the matrix

$$(7.1) \quad \begin{array}{|c|c|} \hline \bar{A}_{12} & \bar{B}_1 \\ \hline \bar{C}_2 & \bar{d} \\ \hline \end{array}$$

of the schema (5.3) has either the schematic form

$$(7.2) \quad \begin{array}{|c|c|} \hline & \oplus \\ & \vdots \\ & \oplus \\ \hline \oplus & \dots & \oplus & * \\ \hline \end{array}$$

or the schematic form

$$(7.3) \quad \begin{array}{|c|c|} \hline \ominus & \oplus \\ \vdots & \vdots \\ \ominus & \oplus \\ \hline - & \end{array}$$

where each \oplus denotes an entry positive or zero, each \ominus an entry negative or zero, and $-$ a negative entry. The \oplus -row and \oplus -column in (7.2) determine optimal extreme points of $P \cap R$ and $Q \cap R$, the corner entry $*$ being the common minimum and maximum value. In (7.3) the \oplus -column determines an extreme point \bar{q} of $Q \cap R$ and the \ominus -column determines the direction of an extreme ray of $Q \cap R$ issuing from \bar{q} , along which the objective function $w \rightarrow +\infty$ because of the corresponding minus entry at the bottom. At the same time the $(\ominus, -)$ -column in (7.3) shows that $P \cap R$ is vacuous and the Dual Program is infeasible.

If $AX = B$ is a consistent system having no solution $X \geq 0$, so that Q exists but $Q \cap R$ is vacuous and the Primal Program (4.2) is infeasible, then it can be shown that there exists a finite succession of elementary pivot transformations terminating in a canonical representation for which the matrix (7.1) of the schema (5.3) has either the schematic form

(7.4)

\oplus	...	\oplus	-
\oplus	...	\oplus	

or the schematic form

(7.5)

		\ominus			
		\vdots			
\oplus	...	0	...	\oplus	-
		\vdots			
		\ominus			
		-			

In (7.4) the \oplus -row at the bottom determines an extreme point \bar{p} of $P \cap R$ and the other \oplus -row determines the direction of an extreme ray of $P \cap R$ issuing from \bar{p} , along which the objective function $\omega \rightarrow -\infty$ because of the corresponding minus entry at the right. In (7.5) the nonpositive column with negative entry at bottom shows that $P \cap R$ is vacuous and the Dual Program is infeasible. The $(\oplus, -)$ -row in (7.4) and (7.5) confirms that $Q \cap R$ is vacuous and the Primal Program is infeasible.

In summary, the terminal possibilities for the simplex method are, in the format of this paper:

Form (7.2) - Primal feasible ($Q \cap R \neq \emptyset$),

Dual feasible ($P \cap R \neq \emptyset$).

Form (7.3) - Primal feasible ($Q \cap R \neq \emptyset$),

Dual infeasible ($P \cap R = \emptyset$).

Form (7.4) - Primal infeasible ($Q \cap R = \emptyset$),

Dual feasible ($P \cap R \neq \emptyset$).

Form (7.5) - Primal infeasible ($Q \cap R \neq \emptyset$),

Dual feasible ($P \cap R = \emptyset$).

From any initial presentation (4.1) of a pair of linear programs (4.2) and (4.3), provided $AX = B$ is a consistent system of linear equations (so that Q exists), it is possible through a finite succession of elementary pivot transformations to reach a terminal canonical representation for which the matrix (7.1) of the schema (5.3) has one of the above four forms (7.2), (7.3), (7.4), (7.5).

8. HOMOGENEOUS LINEAR PROGRAMS AND TRANSPOSITION— DUALITY THEOREMS

In the pair of linear programs (4.2) and (4.3) take $B = 0$ and $d = 0$ to get a homogeneous linear program,

(8.1) Minimize CX constrained by $AX = 0, X \geq 0$,

and its dual program,

(8.2) Solve $UA + C \geq 0$.

(Here it seems convenient to minimize $CX = -w$ rather than to maximize $w = -CX$, to replace the parametric Λ by U , and to omit Ξ .) The programs (8.1) and (8.2) are jointly exhibited by the schema.

$$(8.3) \quad \begin{array}{c} X \\ \boxed{\begin{array}{c} A \\ \hline C \end{array}} \\ \geq 0 \end{array} \quad \begin{array}{l} (\geq 0) \\ = 0 \\ = \min \end{array}$$

The homogeneous linear program (8.1) is clearly feasible, since $X = 0$ satisfies $AX = 0$. There are just two possibilities (corresponding to the two cases set forth in the first paragraph of Sec. 7): Either CX has a zero minimum and (8.2) is feasible or CX is unbounded below for feasible X and (8.2) is infeasible. These two possibilities establish a "theorem of alternatives" for a homogeneous linear program (8.1) and its dual (8.2):

Theorem 1. Either $UA + C \geq 0$ for some U or $CX < 0$ for some $X \geq 0$ such that $AX = 0$ (but not both).

This theorem can be regarded as a fundamental existence theorem for an arbitrary system $UA + C \geq 0$ of nonhomogeneous linear inequalities:

Theorem 2. $UA + C \geq 0$ for some U if, and only if, there is no $X \geq 0$ for which $AX = 0$ and $CX < 0$.

Take $C < 0$. Then $UA + C \geq 0$ implies $UA \geq -C > 0$. Also, $CX < 0$ for $X \geq 0$ if, and only if, $X \neq 0$. Hence, Theorem 1 yields the following classical theorem of Gordan (and later Stiemke), which seems to have been the earliest known Transposition Duality Theorem (see [6]):

Theorem 3. $AX = 0$ for some $X \geq 0$ (i.e., $X \geq 0$ and $\neq 0)$ if, and only if, $UA > 0$ for no U .

Now form the schema

$$(8.4) \quad \begin{array}{cc} & \begin{array}{cc} x_0 & X' \end{array} \quad (\geq 0) \\ \begin{array}{c} U \\ 1 \end{array} & \boxed{\begin{array}{cc} -B & A \\ - & - \\ -1 & 0 \end{array}} & \begin{array}{c} = 0 \\ = \min \end{array} \\ & \begin{array}{cc} \geq 0 & \geq 0 \end{array} \end{array}$$

where A is a matrix, and $-B$ an additional column. Clearly $-UB - 1 \geq 0$ implies $UB \leq -1 < 0$, and $-Bx_0 + AX' = 0$ for $x_0 > 0$, $X' \geq 0$ implies $AX = B$ for $X = (X'/x_0) \geq 0$. Hence the alternatives of Theorem 1, applied to (8.4), establish the following classical theorem of Farkas concerning "convex linear dependence":

Theorem 4. If $UB \geq 0$ for all U such that $UA \geq 0$, then $B = AX$ for some $X \geq 0$ (and conversely).

Next form the schema

$$(8.5) \quad \begin{array}{c} U \\ 1 \end{array} \begin{array}{c} x_1 \quad x_2 \quad x_3^+ \quad x_3^- \\ \boxed{\begin{array}{cc|cc} A_1 & A_2 & A_3 & -A_3 \\ -1 & 0 & 0 & 0 \end{array}} \end{array} \begin{array}{l} = 0 \\ = \min \end{array}$$

$$\begin{array}{cccc} \geq 0 & \geq 0 & \geq 0 & \geq 0 \end{array}$$

where -1 denotes a row of -1 's. Observe that $UA_1 \geq 1$ implies $UA_1 > 0$ and that $UA_3 \geq 0$, $-UA_3 \geq 0$ implies $UA_3 = 0$. Let $x_3 = x_3^+ - x_3^-$. Then Theorem 1, applied to (8.5) establishes the general transposition theorem of T. S. Motzkin:

Theorem 5. Either $UA_1 > 0$, $UA_2 \geq 0$, $UA_3 = 0$ for some U or $A_1x_1 + A_2x_2 + A_3x_3 = 0$ for some $x_1 \geq 0$, $x_2 \geq 0$, x_3 unrestricted.

9. THEOREMS FOR SKEW AND DUAL LINEAR SYSTEMS

Let K be a skew-symmetric (square) matrix, i.e., $K^T = -K$, and I the identity matrix of equal order. Form the homogeneous linear program and its dual:

$$(9.1) \quad \begin{array}{c} U \\ 1 \end{array} \begin{array}{c} X \quad Y \quad Z \quad (\geq 0) \\ \boxed{\begin{array}{cc|cc} K+I & K & I & \\ -1 & 0 & 0 & \end{array}} \end{array} \begin{array}{l} = 0 \\ = \min \end{array}$$

$$\begin{array}{ccc} \geq 0 & \geq 0 & \geq 0 \end{array}$$

where -1 denotes a row of -1 's. Premultiply

$$(K + I)X + KY + IZ = 0$$

by $(X + Y)^T$ to get

$$(X + Y)^T K(X + Y) + (X + Y)^T I(X + Z) = 0.$$

Then, since $(X + Y)^T K(X + Y) = 0$,

$$X^T X + X^T Z + Y^T X + Y^T Z = 0.$$

However, this holds for $X \geq 0$, $Y \geq 0$, $Z \geq 0$ if, and only if, each term is zero; and $X^T X = 0$ if, and only if, $X = 0$. Hence, the homogeneous linear program specified by the rows of (9.1) has a zero minimum and the dual program specified by the columns of (9.1) is feasible. That is, there exists some U^* satisfying the column inequalities of (9.1):

$$U(K + I) \geq 1 > 0, \quad UK \geq 0, \quad UI \geq 0.$$

This establishes the following "skew-symmetric matrix theorem" (see [6], Theorem 5):

Theorem 6. The system $UK \geq 0$ of homogeneous linear inequalities, where $K^T = -K$, possesses a solution $U^* \geq 0$ such that $U^* + U^* K > 0$.

Apply Theorem 6 to the matrix

$$K = \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix}.$$

Then the inequality

$$[\varepsilon, Y^T] \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix} \geq 0$$

possesses a solution $\varepsilon^* \geq 0, Y^* \geq 0$ such that

$$[\varepsilon^*, Y^{*T}] + [\varepsilon^*, Y^{*T}] \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix} > 0.$$

This establishes the following theorem (see [6], Theorem 3) concerning the dual linear systems of schema (2.1) in Sec. 2:

Theorem 7. The column and row equation systems of the schema

$$\begin{array}{c} -Y \\ \varepsilon \begin{bmatrix} A \end{bmatrix} = X \\ = H \end{array}$$

possess solutions $\varepsilon^* \geq 0, H^* \geq 0$ and $X^* \geq 0, Y^* \geq 0$ such that

$$\varepsilon^* + X^{*T} > 0 \text{ and } H^* + Y^{*T} > 0.$$

Apply Theorem 7 to

$$\begin{array}{rcccl}
 & -Y_1 & -Y_2^+ & -Y_2^- & \\
 \begin{array}{l} \Xi_1 \\ \Xi_2^+ \\ \Xi_2^- \end{array} & \boxed{\begin{array}{ccc} A_{11} & A_{12} & -A_{12} \\ A_{21} & A_{22} & -A_{22} \\ -A_{21} & -A_{22} & A_{22} \end{array}} & = & \begin{array}{l} X_1 \\ X_2^+ \\ X_2^- \end{array} \\
 & = 1 & = 2^+ & = 2^- &
 \end{array}$$

where A_{11} is an arbitrary submatrix of a matrix A and A_{12} , A_{21} , A_{22} are the remaining submatrices. Then there exist nonnegative solutions (starred) of the column and row equation systems of (9.2) such that

$$[\Xi_1^*, \Xi_2^{+*}, \Xi_2^{-*}] + [X_1^{*T}, X_2^{+*T}, X_2^{-*T}] > 0$$

and

$$[H_1^*, H_2^{+*}, H_2^{-*}] + [Y_1^{*T}, Y_2^{+*T}, Y_2^{-*T}] > 0.$$

Since the sum of the last two columns of (9.2) is zero, and also the sum of the last two rows,

$$H_2^{+*} + H_2^{-*} = 0 \quad \text{and} \quad X_2^{+*} + X_2^{-*} = 0.$$

Hence H_2^{+*} , H_2^{-*} and X_2^{+*} , X_2^{-*} , being nonnegative, are all zero. Now set

$$\begin{array}{l}
 \text{and} \quad \Xi_2 = \Xi_2^+ - \Xi_2^-, \quad H_2 = H_2^+ - H_2^- \\
 X_2 = X_2^+ - X_2^-, \quad Y_2 = Y_2^+ - Y_2^-
 \end{array}$$

to obtain the following general transposition duality theorem for dual linear systems (see [6], Theorem 6):

Theorem 8. The column and row equation systems of the schema

$$\begin{array}{cc} & \begin{array}{cc} -Y_1 & -Y_2 \end{array} \\ \begin{array}{c} \varepsilon_1 \\ \varepsilon_2 \end{array} & \begin{array}{|cc|} \hline A_{11} & A_{12} \\ A_{21} & A_{22} \\ \hline \end{array} & \begin{array}{c} = X_1 \\ = X_2 \end{array} \\ & \begin{array}{cc} = H_1 & = H_2 \end{array} \end{array}$$

possess solutions

$$\varepsilon_1^* \geq 0, \varepsilon_2^* \leq 0, H_1^* \geq 0, H_2^* = 0$$

and

$$X_1^* \geq 0, X_2^* = 0, Y_1^* \geq 0, Y_2^* \leq 0$$

such that

$$\varepsilon_1^* + X_1^{*T} > 0 \quad \underline{\text{and}} \quad H_1^* + Y_1^{*T} > 0 .$$

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